Curvature and Creases: A Primer on Paper

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***Abstract*—This paper presents fundamental results about how zero-curvature (paper) surfaces behave near creases and apices of cones. These entities are natural generalizations of the edges and vertices of piecewise-planar surfaces. Consequently, paper surfaces may furnish a richer and yet still tractable class of surfaces for computer-aided design and computer graphics applications than do polyhedral surfaces.**

Major portions of this paper are dedicated to exploring issues of curvature definition, convexity, and concavity, and interrelationships among angles associated with creases and generalized vertices and the orientations of associated surfaces in their vicinities. An electrical network representation is suggested in which there flow currents that are analogous to curvature components on the surface.

***Index Terms*—Computer-aided design, computer graphics, developable surfaces, Gaussian curvature, scene analysis.**

INTRODUCTION

OBJECTS bounded by planes were reasonable ones upon which to do initial research in scene analysis. (For an overview of recent research in this area see [1] and [2] in which the work of Clowes, Guzmán, Huffman, Waltz, and others is summarized.) Similarly, piecewise-planar surfaces were simple approximants to more complex surfaces for computer-aided design and computer graphics applications. Polyhedra presented researchers with potentially complex, but yet not totally arbitrary, surfaces, and a corresponding opportunity to develop intuition and analytic techniques. (The reader is referred to [3] for a representative sample of other recent theoretical and applied work on surfaces.)

It is unlikely that one can say much of practical value about surfaces of complete generality. No two neighboring points on an arbitrary surface need have the same tangent plane. By contrast, all points on a plane surface have the same tangent plane. On a developable (or "paper") surface, on the other hand, all points on a given line embedded in the surface have the same tangent plane. That is, the neighborhood of a point on a paper surface can be characterized by a single-parameter family of tangent planes. The author proposes that a paper surface offers a complexity that is, therefore, in a very real sense exactly midway between that of a completely general surface and that of a plane surface. Consequently, paper surfaces constitute a class that may be ideally suited to be both richer than that of plane surfaces and more tractable analytically than that of totally arbitrary surfaces. This paper explores many of the extraordinarily delightful and relatively simple relationships that exist among the quantities associated with a typical point on a paper surface.

Throughout this paper there is exploited a basic fact about a developable surface: the Gaussian curvature (defined later in this paper) is exactly zero at every point on a developable surface. Special attention is given to the simplest possible class of polyhedral vertices that can be formed on paper surfaces. This simplest type has just four plane sectors (the angles of which sum to 360 degrees).

A major part of the paper is dedicated to derivation of equations that express relationships among the sector angles for the basic degree-4 vertex, the associated dihedral angles between the plane sectors, and the angles between these sectors and the osculating plane associated with the vertex. A single parameter that expresses the amount of "flexing" near the vertex is defined. An electrical network representation is developed in which the currents are analogous to the curvature components near the vertex. Finally, the understanding of the basic degree-4 vertex is utilized to develop an understanding of issues of curvature and convexity in the vicinity of apices of arbitrary cones and of points on arbitrary creases on paper surfaces.

THE GAUSSIAN SPHERE AND GAUSSIAN CURVATURE

In this section we review briefly the concept of Gaussian curvature at a point on a surface. An excellent and more detailed treatment of the subject is given in the book by Hilbert and Cohn-Vossen [5].

Consider a closed contour k oriented, say, clockwise and enclosing the given point on the surface. Each point on the contour has an associated unit length vector normal to the surface and oriented away from the surface. Let this set of normal vectors be transferred so that the spatial direction of each is preserved and so that each starts at the center of a unit radius sphere (the Gaussian sphere) and ends on the surface of that sphere. The ratio K of the area G enclosed by the resulting closed contour k' (which we shall call here the *trace* of the contour k) on the sphere to the area F enclosed by the contour k has a definite limit as k shrinks to the given point. This ratio is defined to be the *Gaussian* (or total) *curvature* at the point. When k' is oriented clockwise, the curvature is positive. When k' is oriented counterclockwise, the curvature is negative. The net area enclosed by a trace k' that is more complex is

Manuscript received June 17, 1975; revised October 20, 1975. The research reported in this paper was supported by the National Science Foundation under Grant GJ-28451.

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considered positive when it encloses more area clockwise than it encloses counterclockwise, and negative if more area is enclosed counterclockwise than clockwise.

It is well known that at a given point the Gaussian curvature K is equal to the product of the two principal curvatures at that point. Each of these latter curvatures is equal to the reciprocal of a corresponding one of the two principal radii of curvature, r_1 and r_2. As examples the reader should consider the Gaussian curvature at typical points:

1) on the outside of a sphere of radius r (the two radii of curvature have the same positive values and hence $K = 1/r^2$),
2) on the inside of a spherical surface of radius r (the two radii of curvature have the same negative value and hence again $K = 1/r^2$),
3) on an ellipsoid (where r_1 and r_2 are both positive and $K = 1/r_1r_2$),
4) on a saddle surface, such as a hyperbolic paraboloid (where r_1 and r_2 have opposite signs and K is therefore negative),
5) on a right circular cylinder (where one of the radii is infinite and hence $K = 0$, and
6) at the apex of a cone (where each radius is zero and consequently K is infinite).

The General Polyhedral Vertex

We next consider the representation on the Gaussian sphere of the plane sectors associated with an arbitrary polyhedral vertex. As a specific example, consider the corner of a cube [see Fig. 1(a)] having the three associated planes A, B, and C, and some (it makes no difference which) contour enclosing the vertex clockwise. The $+$ symbols are labels indicating that the corresponding edges are convex, rather than concave. It is clear that the trace consists of three segments, each a great circular arc $\pi/2$ units in length, and encloses an area on the sphere equal to one eighth of the total surface area: $(1/8)(4\pi) = \pi/2$. We note also that the trace makes three abrupt turns to the right, each corresponding to the angle $\pi/2$ associated with each of the sectors. The angles A, B, and C equal both the exterior angles of the spherical triangle and the sector angles on the cube. (We shall use a given capital letter to refer both to a surface plane and to the corresponding sector angle, using context to make the distinction clear.) Each segment of the trace has a length equal to the dihedral angle between the corresponding two planes on the cube. A given segment of trace represents the set of normals associated with all of the planes tangent to the surface and containing the associated edge of the surface.

Another example of a polyhedral vertex is given in Fig. 2. It has two concave ($-$) as well as two convex ($+$) edges. The configuration is to approximate a saddle surface. This requires that the four sector angles, A, B, C, and D (all

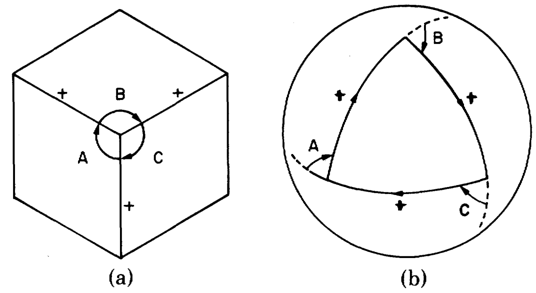


Fig. 1. Representation of a simple polyhedral vertex. (a) Contour around a cube vertex and (b) corresponding trace on the Gaussian sphere.

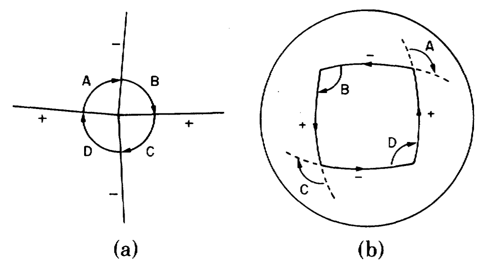
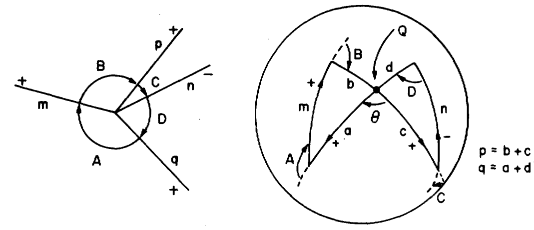


Fig. 2. Representation of another polyhedral vertex. (a) Contour around a "saddle point" vertex and (b) corresponding trace on the Gaussian sphere.

assumed here to be equal), must have a sum that exceeds 2π and thus each sector angle must exceed $\pi/2$. Again the trace on the sphere does not depend on exactly which contour we choose, but only upon the sector angles and the dihedral angles between the pairs of adjacent planes. We show a case in which these dihedral angles are all the same, although that is not a necessity. Note that the orientation of those parts of the trace between planes A and B and between planes C and D is opposite to the orientation of the surface contour. This occurs because of the concavity of the corresponding edges. The area within the trace can be considered to be negative because that area is enclosed counterclockwise and it remains the same even as the contour shrinks toward the vertex. The magnitude of the curvature at the vertex is, therefore, infinite.

In examining the traces of both Fig. 1(b) and Fig. 2(b) we see that the angle by which the trace changes its direction corresponds to the angle of the associated plane sector of the surface at the vertex. This turning is clockwise in all cases (since the corresponding turning of the contour on the surface was assumed always to be clockwise) even though the trace itself may progress counterclockwise. A classic theorem in spherical trigonometry states that *the area of a spherical triangle on a unit radius sphere is equal to the sum of the interior angles minus* π. This quantity is often called the "excess angle." By decomposing the area within a trace on the Gaussian sphere into triangles so that this theorem can be applied it is easy to show that *the area within the trace is equal to* 2π *minus the sum of the sector angles* on the corresponding surface. If the sum of these sector angles is less than 2π the area within the trace is positive [as in Fig. 1(b)]. If the sum of the sector angles is more than 2π the area within the trace is negative [as in Fig. 2(b)].

Except for the case in which the sum of the sector angles at a vertex is 2π, we can then consider that the curvature at a polyhedral vertex is an impulse, the weight of which is equal to the net area enclosed by the corresponding trace. An important observation is that this result is independent of the dihedral angles between the plane sectors.



(a) (b)

Fig. 3. Basic degree-4 polyhedral vertex on a zero-curvature surface. (a) Vertex configuration and (b) trace on the Gaussian sphere.

The Simplest Zero-Curvature Polyhedral Vertex

An especially important subclass of polyhedral vertices is that for which the sum of the sector angles is 2π. This constraint applies to each point on an idealized paper surface. (We consider only regions on such surfaces away from the boundary edges and on which no cutting and rejoining has taken place.)

The simplest way that a trace on the Gaussian sphere can correspond to a polyhedral vertex and yet enclose zero net area is illustrated in Fig. 3(b). Four sectors are easily seen to be necessary because three sectors can only lead to a single triangle. Each degree-4 polyhedral vertex on a zero curvature surface either has three convex and one concave edge (as shown in our example) or has three concave and one convex edge. The latter type of vertex is essentially the mirror image of the type we shall consider as our standard for expository purposes.

The left portion of the trace encloses a triangle having a positive area that is of exactly the same magnitude as that of the triangle enclosed by the right portion of the trace. The symbols m, n, p, and q refer to the magnitude of the dihedral angles between the plane-pairs AB, CD, BC, and DA, respectively.

The arc from B to C in the spherical representation represents the normals to planes that are tangent to the surface along the edge between these two planes. Similarly, the arc from D to A represents the normals to planes that are tangent to the surface along the edge between the latter two planes. The angle θ is the angle between these two edges. The point Q represents the normal to the plane containing both these edges. The magnitudes of the (dihedral) angles between the plane Q and the planes A, B, C, and D are designated a, b, c, and d, respectively, as shown in the figure. It is apparent that $p = b + c$, and $q = a + d$.

Imagine now that the surface pictured in Fig. 3(a) is flattened out so that all dihedral angles are zero (but so that the sector angles remain constant). In that case the common area E of each of the two triangles would become zero (since all normal vectors would have the same spatial orientation). The angle θ would then become equal to $C + D = 2\pi - (A + B)$. More generally (when the paper surface is not flattened) the angle θ between the two edges would be less than that amount.

We observe from Fig. 3(b) that the excess angle in the left and right triangles is $(\pi - A) + (\pi - B) + (\pi - \theta) - \pi = (2\pi - (A + B)) - \theta$ and $C + D + (\pi - \theta) - \pi = (C + D) - \theta$, respectively. Each of these is equal to the common area, E, of the two triangles. We thus conclude that the area of each of the triangles is equal to the decrease in angle between the edges common to B and C and common to D and A as the configuration is flexed from its flattened state to that corresponding to the originally specified dihedral angles. Therefore E can be considered to be a useful parameter that indicates the amount of flexing at the vertex.

Another possible parameter that indicates the amount of flexing at the vertex is obtained by first considering a sheet of paper ruled with a regular pattern of lines obtained by iterating the basic configuration [see Fig. 4(a)]. This type of iteration is possible for any set of sector angles that sum to 2π. The four sector angles are shown at each of the vertices. All edges between sectors marked C and D are to be concave; all others are to be convex. We note also that the vertices are in two classes and that the ruling remains the same when the paper is rotated by π.

Now assume that each edge is given its originally specified dihedral angle (m, n, p, or q). This is possible to do in a consistent way because the opposite ends of any line are associated with the same pair of planes from the configuration around the basic vertex. Consider now, for example, the normals to the sequence of planes $P_1, P_2, \cdots, P_7$. It is clear that the angles between P_1 and P_3, P_3 and P_5, P_5 and P_7, and P_7 and P_9 are all equal. Similarly, the angles between P_{10} and P_{12}, P_{12} and P_{14}, P_{14} and P_{16}, and P_{16} and P_{18} are all equal. By rotating the pattern and recalling the symmetry noted above we conclude in turn that these two angles must be equal. That is, the angles between pairs of planes in the sequence P_1, P_3, P_5, P_7, P_9; in the sequence P_2, P_4, P_6, P_8; in the sequence $P_{10}, P_{12}, P_{14}, P_{16}, P_{18}$; and in the sequence $P_{11}, P_{13}, P_{15}, P_{17}$ are all equal.

There is only one way that the normal vectors for the planes described above can be placed on the Gaussian sphere: The even-numbered planes and the odd-numbered planes must have normal vectors that lie above and below some great circle (an equator) on two parallel circles of constant latitude, one a given angular distance l above that equator and the other the same distance l below that equator. The situation is represented in Fig. 4(b) (a cylindrical projection with the lines straightened for simplicity of representation). The traces associated with two typical vertices (one from each of the two classes) are shown by heavy lines. Our iterated pattern [Fig. 4(a)] was constructed so that the axis of cylindrical symmetry was vertical, as the reader can verify by doing the folding himself, but of course that is not necessarily the case in the general situation.

Our conclusion from the discussion above is that the midpoints of the four arcs constituting the trace for an arbitrary polyhedral vertex of degree four on a zero-curvature surface must lie on a common great circle. The vectors normal to the four plane sectors lie above or below that circle at a common angular distance l. Thus that angle l is another parameter than can be related to the amount of flexing at the vertex.

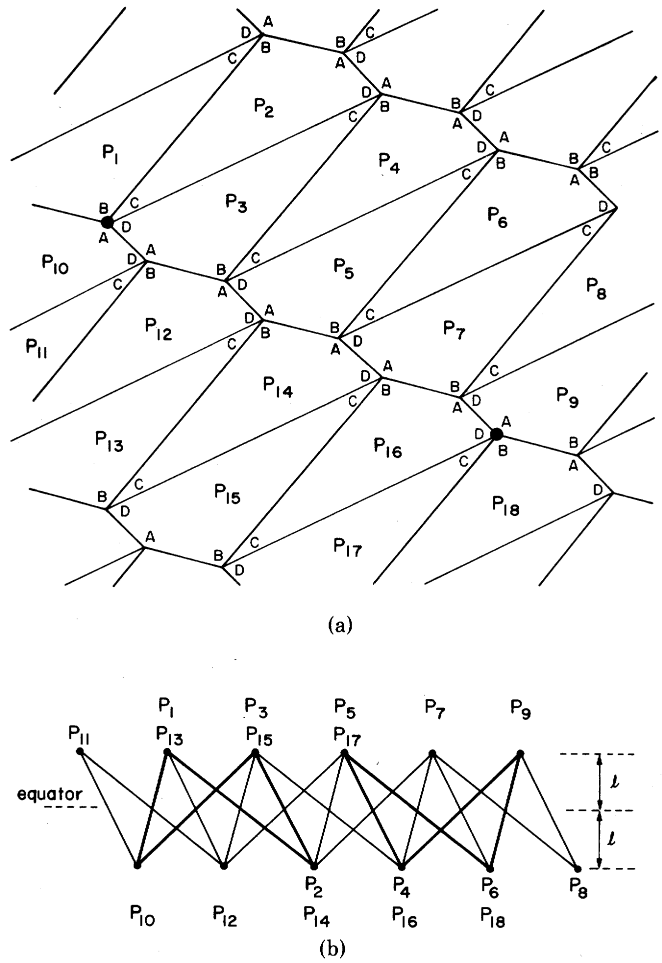


Fig. 4. Paperfolding having cylindrical symmetry. (a) Iterated network and (b) set of traces.

Relationships Among Angles at the Degree-4 Vertex

In this section we derive several basic relationships among the various dihedral and other angles at the general degree-4 polyhedral vertex on a zero-curvature surface. In the diagrams related to these derivations we shall draw the various traces using straight lines. The reader should keep in mind, however, that the segments of the traces shown are actually segments of arcs of great circles on the Gaussian sphere. The appropriate formulas are, therefore, from the realm of spherical trigonometry. In these formulas a segment of arc is measured by the angle between the two radial lines extending from the center of the sphere to the endpoints of the arc. The area of a spherical triangle is itself measured as an angle: the excess angle referred to earlier.

One formula from spherical trigonometry relates the tangent of half the area of a triangle to the tangents of half the two adjacent sides and to the sine and cosine of the angle between these sides. This formula applied to the triangle shown in Fig. 5 yields

$$\tan\frac{E}{2} = \frac{\tan\frac{a}{2}\tan\frac{b}{2}\sin(\pi-\theta)}{1+\tan\frac{a}{2}\tan\frac{b}{2}\cos(\pi-\theta)} = \frac{\tan\frac{c}{2}\tan\frac{d}{2}\sin(\pi-\theta)}{1+\tan\frac{c}{2}\tan\frac{d}{2}\cos(\pi-\theta)}.$$

It can be derived in a straightforward way that

$$\tan\frac{a}{2}\tan\frac{b}{2} = \tan\frac{c}{2}\tan\frac{d}{2}. \tag{1}$$

A "cosine law" of spherical trigonometry relates the cosine of one (interior) angle of a triangle to the sine and cosine of the other two angles and to the cosine of the side opposite the first angle. This formula applied to the triangles of Fig. 5 yields

$$\cos(\pi-\theta) = -\cos(\pi-A)\cos(\pi-B) + \sin(\pi-A)\sin(\pi-B)\cos m$$

and

$$\cos(\pi-\theta) = -\cos C\cos D + \sin C\sin D\cos n.$$

By equating these and by adding $\sin A\sin B + \sin C\sin D$ to each side

$$\begin{aligned}&(\sin A\sin B - \cos A\cos B)\\ &\quad + \sin A\sin B\cos m + \sin C\sin D\\ &\quad = (\sin C\sin D - \cos C\cos D)\\ &\quad + \sin C\sin D\cos n + \sin A\sin B.\end{aligned}$$

Since $A + B + C + D = \pi$ the two parenthesized expressions are each equal to $-\cos(A+B) = -\cos(C+D)$. It follows that

$$\sin A \sin B\,(1 - \cos m) = \sin C \sin D\,(1 - \cos n).$$

Because $2\sin^2\alpha = 1 - \cos 2\alpha$ for any angle α, we conclude that

$$\frac{\sin^2 \frac{m}{2}}{\sin^2 \frac{n}{2}} = \frac{\sin C \sin D}{\sin A \sin B}. \tag{2a}$$

By extending the parts of the trace associated with the dihedral angles m and n, we obtain the construction shown in Fig. 5. The two large triangles have the same area and, therefore, we can apply the cosine law to these two triangles in the same way that we did in the above derivation. A derivation of the same style leads us to conclude that

$$\frac{\sin^2 \frac{p}{2}}{\sin^2 \frac{q}{2}} = \frac{\sin A \sin D}{\sin B \sin C}. \tag{2b}$$

A more difficult derivation, not recorded here, allows us to establish that

$$\frac{\sin^2 \frac{u}{2}}{\sin^2 \frac{v}{2}} = \frac{\sin A \sin C}{\sin B \sin D} \tag{2c}$$

where u and v (shown in Fig. 5) are, respectively, the angle between the (normals to the) planes B and D and the angle between the planes A and C.

The symmetry among (2a)–(2c) is apparent. The first of these has an especially interesting interpretation. If a sheet of paper is creased according to the plan illustrated in the diagram of Fig. 3(a), it becomes apparent that as the configuration is flexed either the angle p or the angle q will first become equal to u. In either case we call the state the "binding" of the configuration.[1] Once this state is reached no further flexing is possible that keeps the sectors plane, as is required. Which event occurs first is determined by the ratio $\sin A \sin D$: $\sin B \sin C$. If that ratio is greater than one, the bind first occurs because p becomes equal to π; if it is less than one the bind first occurs because q becomes equal to π. If the ratio equals one both angles become equal to π simultaneously (as do the angles m and n). It can easily be shown that this latter situation occurs only when $A + C = B + D = \pi$.

Because (2a)–(2c) are all especially simple and because each involves the sines of half of two of the dihedral angles, it would be natural to suspect that the relationships between pairs of adjacent dihedral angles (m and p, p and n, etc.) are also simple ones. This, unfortunately, is not the case. By a very difficult derivation the author has proved that, for example,

$$\left[1 \mp \sqrt{\frac{\sin B \sin D}{\sin A \sin C}} \sqrt{\frac{1 - \frac{\sin A \sin D}{\sin B \sin C} \sin^2 \frac{q}{2}}{1 - \sin^2 \frac{q}{2}}}\right] \cdot \left[1 \pm \sqrt{\frac{\sin B \sin D}{\sin A \sin C}} \sqrt{\frac{1 - \frac{\sin C \sin D}{\sin A \sin B} \sin^2 \frac{n}{2}}{1 - \sin^2 \frac{n}{2}}}\right] = 1 - \frac{\sin B \sin D}{\sin A \sin C}. \tag{3}$$

The implications of this and similar equations on the flexings of networks of polyhedral vertices on zero-curvature surfaces will be pursued in a later paper.

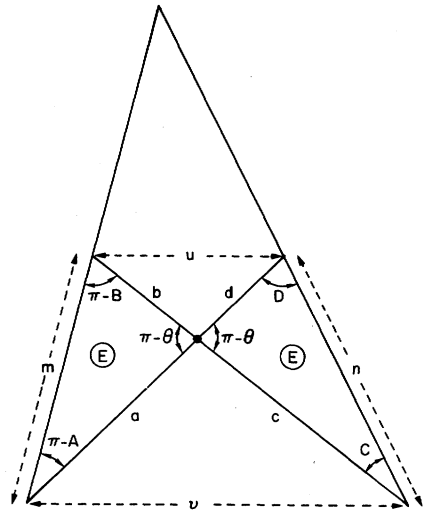


Fig. 5. Quantities related to the derivation of (1)–(3).

An Electrical Network Analogy for a General Polyhedral Vertex on Paper

In Fig. 6(a) is shown an example of a rather complex configuration of lines at a vertex on a paper surface *before* it is flexed; that is, when all the dihedral angles are incremental ones. All of the normal vectors have nearly the same orientation and occupy only a small region on the Gaussian sphere. This small region is nearly a plane and the trace is typified by the one given in Fig. 6(b). (For a more general discussion of the problem of representing the information on the Gaussian sphere on a plane, see [4]). Note that each segment of the trace is perpendicular to the corresponding edge on the surface and that the direction of the trace is reversed when the surface contour traverses a sector that is a boundary between a region of + edges and a region of

[1] The term "binding" is due to Dr. R. Resch of the Department of Computer Science at the University of Utah. Dr. Resch is a computer-artist whose paperfoldings are well known.

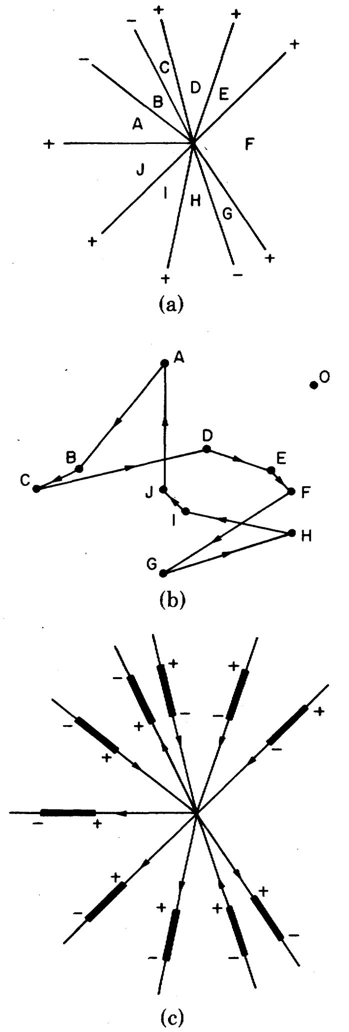


Fig. 6. Electrical network analog for a zero-curvature polyhedral vertex configuration. (a) Vertex configuration, (b) trace, and (c) analogous network.

− edges. When the various dihedral angles are larger the trace covers more of the Gaussian sphere, always however in such a way that the angles between successive segments remain constant. As long as these angles sum to 2π the net area within the trace will be zero.

The area within the trace may also be computed to be zero by summing the areas of a set of triangles, each corresponding to a segment of the trace. Consider, for instance, an arbitrarily placed origin such as the one designated by O in Fig. 6(b). This origin would ordinarily be the normal to the picture plane upon which the vertex configuration is projected. One of the components of the sum is the area of the triangle OAB. This area is considered negative because the directed segment AB has a counterclockwise orientation with respect to O. Another component is the area of the triangle OFG. This area is positive since the directed segment FG has a clockwise orientation with respect to O. The sum of such areas is zero regardless of the position of the origin.

When all dihedral angles are incremental (the only case to be considered here) the area of a component triangle is proportional to the base of the triangle times its height. For instance, the area of the triangle ODE is proportional to the product of the length of DE by the distance from the origin, O, to that segment. The former is the dihedral angle along the edge common to planes D and E. The latter can be interpreted to be the tangent of the angle between that edge and the reference (picture) plane. (See [4] for a detailed explanation of these results and some of those that immediately follow.) If the dihedral angle is divided by a given quantity and the tangent is multiplied by that same quantity the product is unaltered. It is convenient to choose that quantity to be the projected length of the edge upon the picture plane because the tangent times this length is equal to the change in range (measured, say, from the camera) along that edge.

Because change in range to various points on a surface is a potential-like quantity (for instance, the sum of these changes around a closed contour on a surface is zero) the following electrical network analogy suggests itself. (See Fig. 6(c); the broad line segments represent resistors.) Let each edge on the zero-curvature surface correspond to a resistor. The voltage across the resistor will be the change in range along the corresponding edge. The conductance of the resistor will be the dihedral angle associated with that edge divided by the projected length of the edge. The sign of that conductance is the label (+ or −) that is associated with the edge.

For our example the form of the network and the current and voltage directions corresponding to the given position of the origin are shown in Fig. 6(c). Note especially that with the conventions given above for the signs of the voltage and conductance for each element, the resulting component of current flow will be away from or toward the node depending on whether the corresponding triangular area component is positive or negative, respectively. Because these areas sum to zero, the currents also do. Consequently, in this analogy *currents are proportional to curvature components*. The choice of the origin (that is, the choice of the direction from the surface to the viewer) determines both the conductances and the voltages across these conductances. For any choice of origin, however, the net current at a network node is zero.

Other analogies are of course possible in which dihedral angles, slopes of edges, and components of curvature correspond to currents, resistances or conductances, and voltages in different ways. For the more general case of a vertex at which the curvature is not zero the analogous network would require a current proportional to the area enclosed by the trace to be injected into the node. Finally, the situation in which the surface is not almost flat would require that the appropriate formulas from spherical rather than plane trigonometry be used in calculating the areas. These other analogies and extensions will not be elaborated on in this present paper. The purpose of all such analogies is to present an alternate visualization of the surface.

The General Zero-Curvature Cone

It is well known [6] that a surface having zero curvature contains embedded "generating" lines and that for every point on a given line the surface has the same tangent plane. Thus if at a given point on a surface there is an associated normal vector, that vector is also appropriate at all other points of the same generating line.

The configurations of edges at the polyhedral vertices shown in Figs. 3(a) and 6(a) are special examples of cones. A set of lines that are all incident at the apex of a cone can be embedded in the conical surface. In general, the surface surrounding the apex will be divided into regions that may be classified as convex (+) interleaved with those that may be classified concave (−), as is shown in Fig. 7(a). In our example we show the boundaries between these regions as dashed lines. The convexity or concavity associated with these regions may be distributed over them so that the dihedral angle for a given embedded line may be arbitrarily small.

We assume again that the sheet of paper forming the cone is nearly flat so that all tangent planes have approximately the same orientation. More generally, the trace will cover an appreciable area of the sphere, but the turnings of the trace to the left or right will be the same as are described here.

Consider a circular contour on the surface of the cone, centered at the apex. This contour crosses each of the generating lines at a right angle. Therefore, the instantaneous direction of the trace at a given point is exactly the same as at the corresponding point on the contour if the given point is in a convex region (for instance, point i). For a point (such as j) in a concave region these two directions are exactly opposite to each other. The various cusps indicated by "β" in Fig. 7(b) correspond to the boundaries previously mentioned. The net area enclosed by the trace is, of course, zero. If a noncircular contour is chosen on the surface [for example, the one indicated in Fig. 7(a)] the corresponding trace will still be the one demonstrated since there is a single tangent plane corresponding to all points on a given generating line.

It is apparent that a distributed resistive network associated with the conical surface is possible. For the general case we may expect that the current (component of curvature) that flows along a given generating line will be an incremental one unless that line has a nonincremental dihedral angle.

We note also that in the example of Fig. 7(b) there is a locus of points (shown shaded) such that each point has the following property: the portions of the trace that correspond to convex (+) regions turn clockwise around the point and those that correspond to concave (−) regions

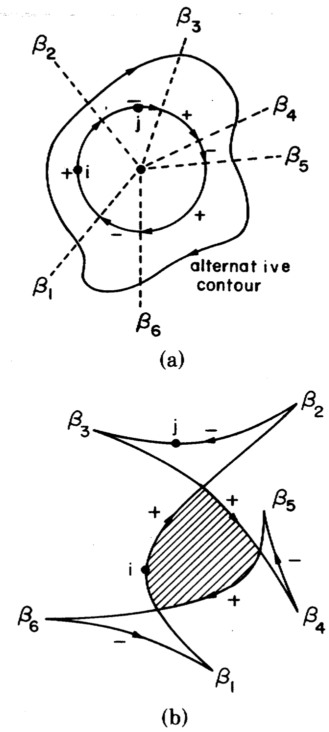


Fig. 7. Cone having distributed curvature. (a) Conical surface and (b) trace.

turn counterclockwise. Such points on the Gaussian sphere correspond to planes that are tangent to the vertex in such a way that the surface near the vertex would be entirely beyond the tangent plane; that is, further away from the viewer. This type of vertex can therefore be classified as "convex."

If we were to consider a related conical surface obtained from the one depicted in Fig. 7(a) by changing the sign of the dihedral angle associated with each generating line, a vertex would result that would be classified as "concave." For a concave vertex there is a locus of points on the Gaussian sphere that is associated with tangent planes that would be beyond the surface. These points on the sphere are passed clockwise by the − portions of the trace and passed counterclockwise by the + portions of the trace.

The traces of Figs. 1(b), 3(b), and 6(b) indicate that the associated vertices are convex. The normals to tangent planes that establish the vertex of Fig. 3(a) as convex are represented inside triangle ABQ. It can be seen that the trace of Fig. 2(b) indicates that there can be no tangent plane entirely on one side or the other of the given surface. That vertex is neither convex nor concave.

It is possible to have vertices that terminate edges on a paper surface. The convexity or concavity of the terminal vertex can make a dramatic difference on the orientation of the nearby surface normals. In Fig. 8(a), for example, we depict a cone associated with a convex vertex; one of the generating lines of the cone is convex and has a nonzero

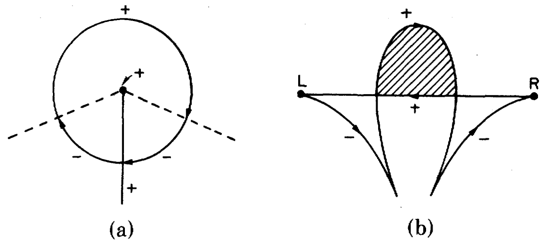


Fig. 8. Convex vertex terminating a convex edge. (a) Conical surface and (b) trace.

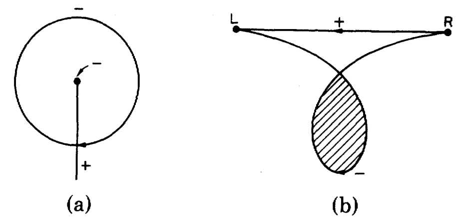


Fig. 9. Concave vertex terminating a convex edge. (a) Conical surface and (b) trace.

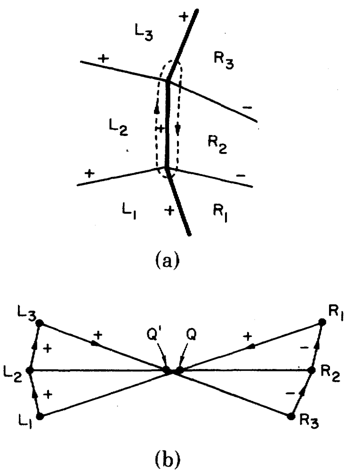


Fig. 10. Approximation to a curved convex crease. (a) Surface and (b) corresponding trace.

dihedral angle. A possible corresponding trace is shown in Fig. 8(b). The points L and R correspond to planes that both contain the indicated line. In Fig. 9(a) there is depicted a cone associated with a concave vertex. Again one line is convex and has a nonzero dihedral angle. The trace for this cone is quite different from the preceding one.

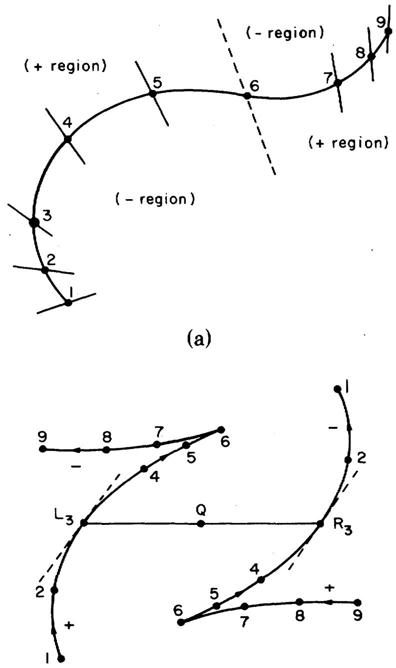
Traces for Surfaces Near Curved Creases

In the preceding section we generalized the concept of a polyhedral vertex to include the apex of an arbitrary cone. In this section we generalize the concept of an edge of a polyhedron to include an arbitrarily curved crease. As we shall see, the fact that the surfaces with which we are dealing have zero curvature places significant constraints on the orientation of the nearby tangent planes.

Surfaces near a portion of a convex crease can be approximated by planes such as those shown in Fig. 10(a). In the corresponding trace of Fig. 10(b) we recall that the points Q and Q' represent the planes determined by portions of the crease (indicated by heavy lines) at the two vertices. These planes correspond to "osculating planes" [7] for the crease. In general, neighboring points on a crease will have different osculating planes. We observe from Fig. 10(b) that *the change in position of the representation of the osculating plane (from Q to Q' along the line L_2R_2) as we move along the crease is orthogonal to the orientation of the corresponding portion of the crease itself.*

We are interested in the trace as the two vertices approach each other and as all three L-planes (and all three R-planes) become the same. The resulting situation is the one that pertains at a single point on a curved crease. The pair of planes L and R are then the planes that contain that point and that are tangent to the surfaces to the left and right of the crease. It is apparent because of the zero-area constraint that in this limiting situation the point Q must be midway between the points L and R. In other words, *the tangent planes L and R make equal angles with the osculating plane Q.*

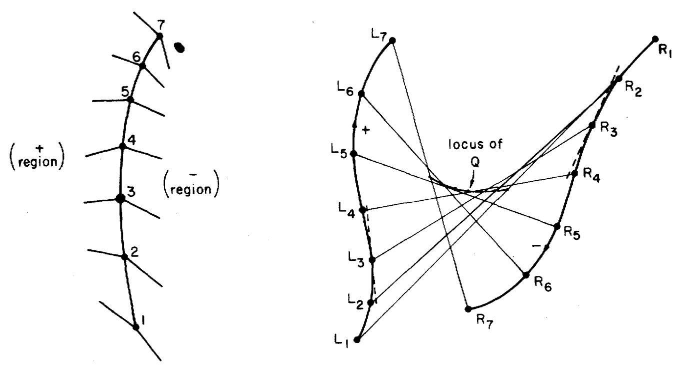
We next consider the implications of this result when it is applied to a crease (see Fig. 11) that is forced to lie in a given plane. That plane is, for this special case, the osculating plane for all points on the crease. Note that the trace consists of two components that have rotational symmetry about the point Q. Observe also that the line L_3QR_3 is perpendicular to the tangent to the crease at the corresponding point. Because of the rotational symmetry, the tangents to the pair of points (for instance, L_3 and R_3) on the trace have the same direction (perpendicular to the corresponding generating lines) and the distances to these tangents from Q must be the same. We conclude that, *for the case of a crease contained in a single osculating plane,*



(a)

(b)

Fig. 11. Representation of a convex crease contained in a single osculating plane. (a) Convex crease with associated generating lines and (b) corresponding trace.



(a) (b)

Fig. 12. Representation of a convex crease with a changing osculating plane. (a) Convex crease with associated generating lines and (b) corresponding trace.

the pair of generating lines at any point on the crease makes equal angles with that plane. That is, the generating lines are reflected from the plane as rays of light would be reflected from a mirror. This is certainly a fundamental and aesthetically pleasing result.

A special case is worthy of separate mention. If all the generating lines (projected onto the single osculating plane) cross the crease at right angles then all portions of the trace [Fig. 11(b)] lie on a single circle. The angular distance from the point Q to any point on the trace is then a constant that depends only upon the amount of flexing along the crease. This angular distance can be interpreted as one half of the dihedral angle between the two tangent planes associated with any point on the crease. This dihedral angle is constant for all points on the crease as long as all portions of the crease lie in the single osculating plane. As the crease is flexed, this common dihedral angle also increases and the radii of curvature associated with all points on the crease are multiplied (decreased) by a common factor. This factor can be proven to be the cosine of one half of the dihedral angle associated with the crease. (A future paper will elaborate on this and other related issues.)

More generally, we can expect that as we move a point along a curved crease the associated osculating plane will change its orientation. (The concept of "torsion" [8] applies to this situation, but an exploration of that concept in the context of zero-curvature surfaces is beyond the scope of this paper.) An example of such a crease is given in Fig. 12. As was noted in the example of Fig. 10, the direction of the change in orientation of Q (as depicted in the representation of the Gaussian sphere) is perpendicular to the tangent to the crease. The pair of tangents to the trace, for example at points #3, need not be parallel and, therefore, the generating lines need not take a common direction in the picture. Nor is it generally necessary that the angles that a pair of generating lines make with respect to their corresponding osculating plane be equal to each other.

Summary

This paper has attempted to place before the reader in a single place a number of the most important facts about how zero-curvature surfaces behave near creases and apices of arbitrary cones. These latter concepts are analogous to those of edges and vertices on objects bounded by plane surfaces. I hope that the reader as a result of my efforts may have begun to appreciate the singular beauty of the relationships I have demonstrated for this more general type of surface. In particular, I suggest that it may be fruitful to visualize components of curvature to be flowing over a surface in such a way that the net flow at any point is equal to the curvature at that point. The zero-curvature surfaces discussed here furnish the simplest examples for this kind of visualization.

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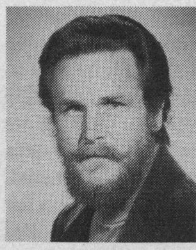
[6] *Ibid,* pp. 204–205.
[7] *Ibid,* pp. 178–180, 205.
[8] *Ibid,* pp. 181–182.



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